CSE 5311: Design and Analysis of Algorithms

Single-Source Shortest Path Algorithms

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Shortest Paths

When you hear the term *shortest path*, you may think of the shortest physical distance between your current location and wherever it is you're going.

Finding the most optimal route via GPS is one of the most widely used mobile applications.

Physical paths are not the only types we may wish to find a shortest path for.

Shortest Paths

Other applications include...

- **Network Routing**: To improve network performance, it is critical to know the shortest path from one system to another in terms of latency.
- **Puzzle Solving**: For puzzles such as a Rubik's cube, the vertices could represents states of the cube and edges could correspond to a single move.
- **Robotics**: Shortest paths in terms of robotics have a lot to do with physical distances, but it could also relate the completing a task efficiently.

Definition

Shortest Paths

Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, a source vertex $s \in V$, and a destination vertex $t \in V$, find the shortest path from s to t.

The weight of a path is defined as the sum of the weights of its edges:

$$w(p) = \sum_{e \in p} w(e).$$

Shortest Paths

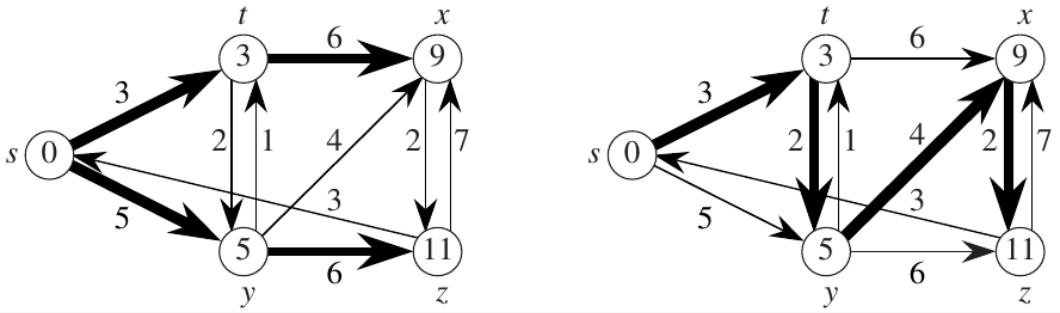
The shortest-path weight between two vertices u and v is given by

$$\delta(u, v) = \begin{cases} \min_{p \in P(u,v)} w(p) & \text{if } P(u, v) \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

where $P(u, v)$ is the set of all paths from u to v.

The shortest-path weight from s to t is given by $\delta(s, t)$.

Shortest Paths



Examples of shortest paths (Cormen et al., 2022).

Shortest Paths

The output of a shortest-path algorithm will produce, for each vertex $v \in V$:

- $v.d$: The shortest-path estimate from s to v.
- $v.\pi$: The predecessor of v in the shortest path from s to v.

Shortest Paths

Shortest-path algorithms have optimal substructure.

Lemma 22.1

Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k. For any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

Shortest Paths

Can a shortest path contain a cycle?

Shortest Paths

Can a shortest path contain a cycle?

No. If so, we could simply traverse the cycle as many times as we wanted to reduce the weight of the path.

For positive-weight cycles, if a shortest path included a cycle, then surely we could remove the cycle to get a lower weight.

Shortest Paths

Some shortest-path algorithms require that the edge weights be strictly positive.

For those that do not, they may have some mechanism for detecting negative-weight cycles.

Shortest Path

As we build a shortest path, we need to keep track of which vertices lead us from the source to the destination.

Some algorithms maintain this by keeping a **predecessor** attribute for each vertex in the path.

Solutions such as the Viterbi algorithm keep an array of indices that correspond to the vertices in the path.

Relaxation

Relaxation

There is one more important property to define before discussing specific algorithms: **relaxation**.

Relaxing an edge (u, v) is to test whether going through vertex u improves the shortest path to v.

If so, we update the shortest-path estimate and predecessor of v to reflect the new shortest path.

Relaxation requires that we maintain the shortest-path estimate and processor for each vertex.

Relaxation

First, the path estimates and predecessor array are initialized.

```
def initialize_single_source(G, s):
    for v in G.V:
        v.d = float('inf')
        v.pi = None
    s.d = 0
```

Relaxation

When the values are changed, we say that the vertex has been **relaxed**.

```
def relax(u, v, w):
    if v.d > u.d + w(u, v):
        v.d = u.d + w(u, v)
        v.pi = u
```

Relaxation

Relaxation has the following properties.

- If $v.d \neq \infty$, then it is always an upper bound on the weight of a shortest path from the source to that vertex.
- $v.d$ will either stay the same or decrease as the algorithm progresses.
- Once $v.d = \delta(s, v)$, it will never change.
- $v.d \geq \delta(s, v)$ always.
- After i iterations of relaxing on all (u, v), if the shortest path to v has i edges, then $v.d = \delta(s, v)$.

The Bellman-Ford Algorithm

Bellman-Ford

The Bellman-Ford algorithm is a dynamic programming algorithm that solves the single-source shortest-paths problem in the general case in which edge weights may be negative.

Bellman-Ford

If a negative-weight cycle is reachable from the source, then the algorithm will report its existence.

Otherwise, it will report the shortest-path weights and predecessors.

It works by relaxing edges, decreasing the shortest-path estimate on the weight of a shortest path from s to each vertex v until it reaches the shortest-path weight.

Bellman-Ford

```
def bellman_ford(G, w, s):
    initialize_single_source(G, s)
    for i in range(1, len(G.V)):
        for (u, v) in G.E:
            relax(u, v, w)
    for (u, v) in G.E:
        if v.d > u.d + w(u, v):
            return False
    return True
```

Example: Bellman-Ford

Bellman-Ford Example

In the first step, the graph is initialized with the source vertex having a distance of 0 and all other vertices having a distance of ∞.

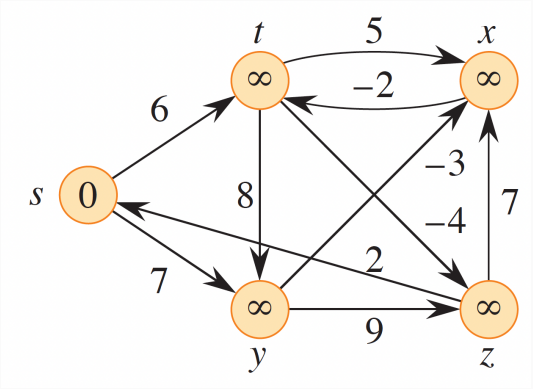


Figure 1: First step of Bellman-Ford (Cormen et al., 2022).

Bellman-Ford Example

The only edges eligible to be relaxed are those connected to the source.

Edges beyond that would not be eligible until t and y are updated.

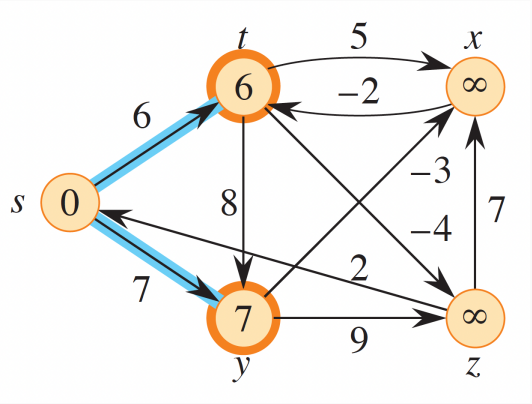


Figure 2: The first edges are relaxed (Cormen et al., 2022).

Bellman-Ford Example

In practice, the inner loop across edges could combine this and the previous step if (s, t) and (s, y) come before (t, x) and (t, y).

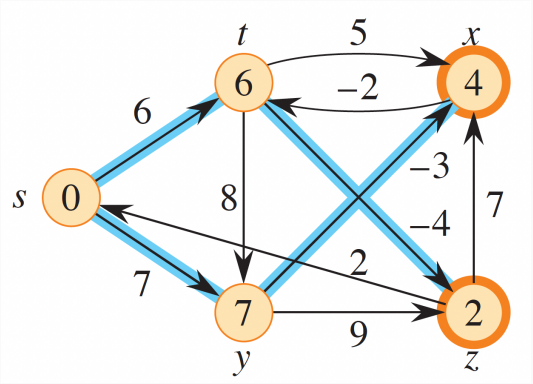


Figure 3: The next eligible edges are relaxed. (Cormen et al., 2022).

Bellman-Ford Example

The negative weight of (x, t) causes the shortest-path estimate of t to decrease.

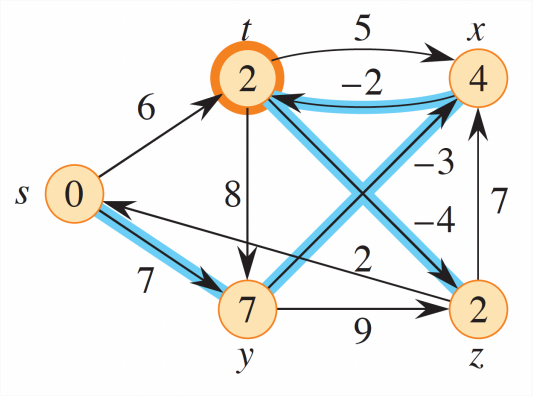


Figure 4: t is updated with new information. (Cormen et al., 2022).

Bellman-Ford Example

In the last check, z is updated with the shortest-path estimate of t.

All vertices have been updated and there are no negative-weight cycles, so the algorithm terminates.

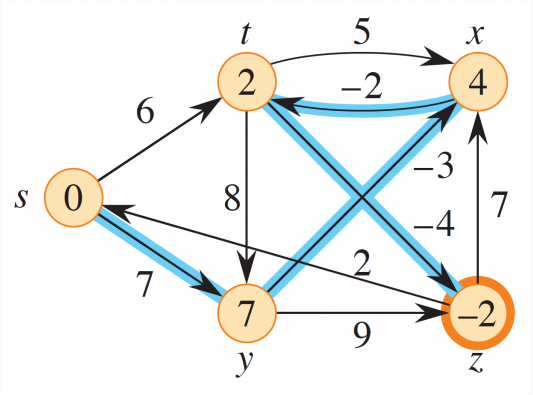


Figure 5: Convergence of Bellman-Ford (Cormen et al., 2022).

Bellman-Ford

A Python implementation is available here.

Correctness

Claim

Bellman-Ford is guaranteed to converge after $|V| - 1$ iterations, assuming no negative-weight cycles.

Correctness

Proof

The first iteration relaxes (v_0, v_1). The second iteration relaxes (v_1, v_2), and so on.

- The **path-relaxation** property from before implies that $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$.
- If there is a negative-weight cycle, then the shortest path to v_k is not well-defined.
- This is verified in the final loop over the edges.

Correctness

Final Check for Negative-Weight Cycles

```
for (u, v) in G.E:
    if v.d > u.d + w(u, v):
        return False
```

Correctness

If there exists a negative-weight cycle $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$ that can be reached from s, then

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0.$$

Correctness

To complete the proof by contradiction, assume that Bellman-Ford returns True .

Then we would have that $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$ by the **triangle inequality** property.

Correctness

If we sum around the cycle, we get

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k} \big(v_{i-1}.d + w(v_{i-1}, v_i)\big) \\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

Correctness

Since the vertices are in a cycle, each vertex appears only once in each summation $\sum_{i=1}^{k} v_i.d$ and $\sum_{i=1}^{k} v_{i-1}.d$.

Subtracting this from both sides of the inequality, we get

$$0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i).$$

Correctness

This contradicts the assumption that there is a negative-weight cycle.

Therefore, if Bellman-Ford returns `True`, then there are no negative-weight cycles. ■

Analysis

Using an adjacency list representation, the runtime of Bellman-Ford is $O(V^2 + VE)$.

The initialization takes $\Theta(V)$.

Each of the $|V| - 1$ iterations over the edges takes $\Theta(V + E)$, and the final check for negative-weight cycles takes $\Theta(V + E)$.

If the number of edges and vertices is such that the number of vertices are a lower bound on the edges, then the runtime is $O(VE)$.

Example

Run Bellman-Ford on the given path using z as the source.

Then change the weight of (z, x) to 4 and run it again with s as the source.

Example

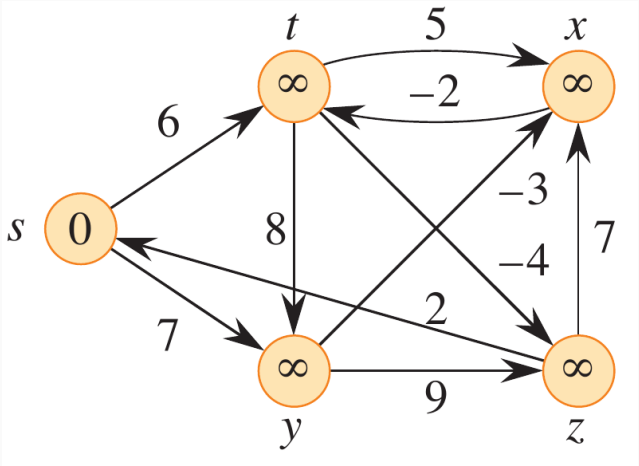


Figure 6: Example of Bellman-Ford (Cormen et al., 2022).

Shortest Paths on a DAG

Shortest Paths on a DAG

If we are given a directed acyclic graph (DAG), we can solve the single-source shortest path problem in $O(V + E)$ time.

By definition, the graph has no cycles and thus no negative-weight cycles.

Shortest Paths on a DAG

```
def dag_shortest_paths(G, w, s):
    initialize_single_source(G, s)
    for u in topological_sort(G):
        for v in G.adj[u]:
            relax(u, v, w)
```

Example

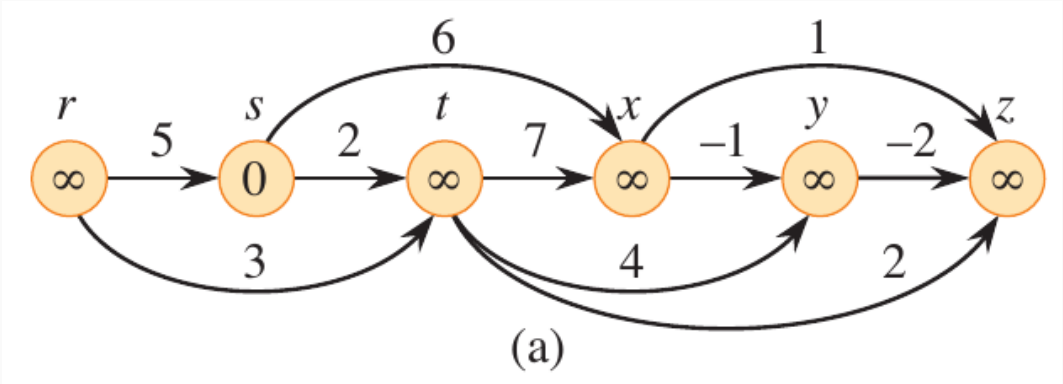


Figure 7: Example of shortest paths on a DAG (Cormen et al., 2022).

Analysis

The runtime of `dag_shortest_paths` is $O(V + E)$, where V is the number of vertices and E is the number of edges.

The topological sort takes $O(V + E)$ time.

Initializing the vertices takes $O(V)$ time.

The first `for` loop makes one iteration per vertex, and the inner loop relaxes each edge only once.

Dijkstra's Algorithm

Dijkstra's Algorithm

Dijkstra's algorithm also solves the single-source shortest path problem on a weighted, directed graph $G = (V, E)$ but requires nonnegative weights on all edges.

Dijkstra's Algorithm

It works in a breadth-first manner.

A minimum priority queue is utilized to keep track of the vertices that have not been visited based on their current minimum shortest-path estimate.

The algorithm works by relaxing edges, decreasing the shortest-path estimate on the weight of a shortest path from s to each vertex v until it reaches the shortest-path weight.

Dijkstra's Algorithm

```
def dijkstra(G, w, s):
    initialize_single_source(G, s)
    S = []
    Q = G.V
    while Q:
        u = extract_min(Q)
        S.append(u)
        for v in G.adj[u]:
            prev_d = v.d
            relax(u, v, w)
            if v.d < prev_d:
                decrease_key(Q, v)
```

Example: Dijkstra's Algorithm

Dijkstra's Algorithm Example

In the first step, the source node is removed from the queue and appended to the set of visited nodes.

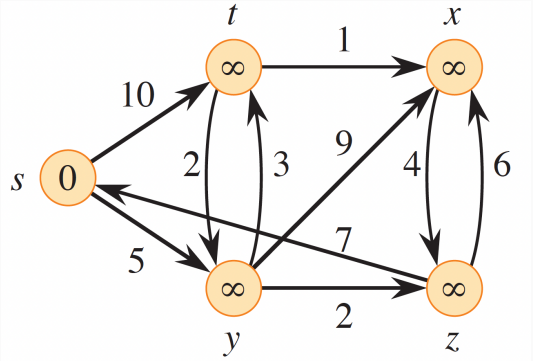


Figure 8: The source node is removed from the queue. (Cormen et al., 2022).

Dijkstra's Algorithm Example

The edges to nodes adjacent to s are relaxed.

The shortest path estimate for t and y is updated.

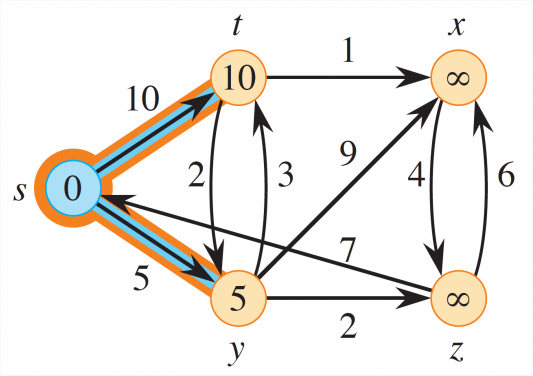


Figure 9: s's neighbors are relaxed. (Cormen et al., 2022).

Dijkstra's Algorithm Example

Node y has the lowest estimate and is removed from the queue.

The edges to t, x, and z are relaxed.

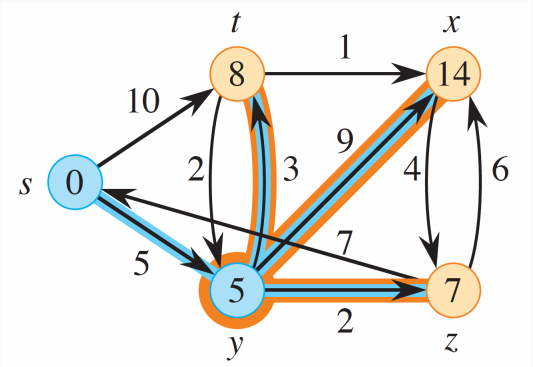


Figure 10: y is added to the set of visited nodes. (Cormen et al., 2022).

Dijkstra's Algorithm Example

Node z has the next lowest estimate and is removed from the queue.

The only eligible edge to relax is (z, x).

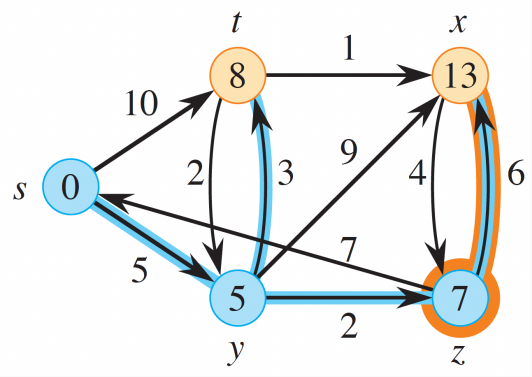


Figure 11: z is added to the set of visited nodes. (Cormen et al., 2022).

Dijkstra's Algorithm Example

Node t has the next lowest estimate with 8 and is removed from the queue.

The shortest-path estimate for its neighbor x is updated from 13 to 9.

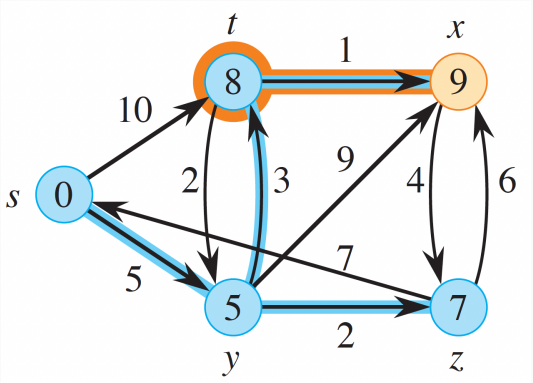


Figure 12: t is added to the set of visited nodes. (Cormen et al., 2022).

Dijkstra's Algorithm Example

Node x is the only remaining node in the queue.

There are no adjacent edges that are eligible to be relaxed.

Since the queue is now empty, the algorithm terminates.

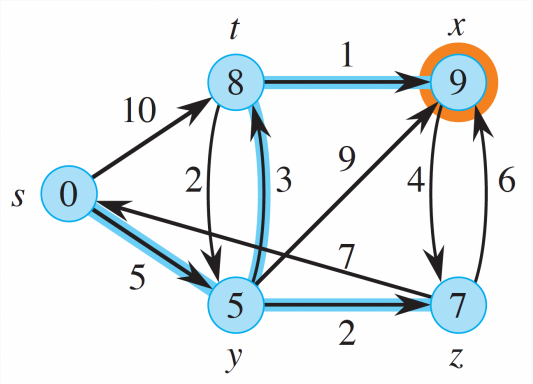


Figure 13: x is added to the set of visited nodes. (Cormen et al., 2022).

Dijkstra's Algorithm

A Python implementation is available here.

Analysis

See Chapter 22 of *Introduction to Algorithms* for a detailed analysis of Dijkstra's algorithm.

Inserting the nodes and then extracting them from the queue yields $O(V \log V)$.

After extracting a node, its edges are iterated with a possible update to the queue. This takes $O(E \log V)$.

The total runtime is $O((V + E) \log V)$.